Algebra

Polynomial

$f(x) = a_0x^n + a_1x^{n-1} + \cdots .. + a_n (a_0 \neq 0)$ is called a polynomial in variable x, where $a_0, a_1, \ldots, a_n$ are real numbers and n is a non-negative integer, is called degree of polynomial.

e.g. Polynomial $(x - a)$ is a degree of 1 and polynomial $x^2 - 7x + 12$ is a degree of 2 .

Note - If $f(x) = 0$, then it is said to be polynomial equation.

Fundamental Operations on Polynomials

Some operations based on polynomials are discussed below

1. **Addition of Polynomials:** Polynomials can be added by arranging their like terms and combining them.
2. **Subtraction of Polynomials:** Polynomials can be subtracted by arranging their like terms and by changing sign of each term of the polynomial to be subtracted and then added.
3. **Multiplication of Polynomials:** We know that, (i) the product of two factors with like signs is positive and product of unlike signs is negative.

   (ii) if x is any variable and m, n are positive integers, then $x^m \times x^n = x^{m+n}$

   Thus, $x^3 \times x^6 = x^{(3+6)} = x^9$.

4. **Division of a Polynomial by a Polynomial**

   The following steps are given below

- Firstly, arrange the terms of the dividend and divisor in descending order of their degrees.
- Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.
- Multiply all the terms of the divisor by the first term of the quotient and subtract the result from the dividend.
- Consider the remainder (if any) as a new dividend and proceed as before.

- Repeat this process till we obtain a remainder which is either 0 or a polynomial of degree less than the degree of the divisor.

Example 1: Find the quotient and the remainder when $x^4 + 1$ is divide by $x - 1$.

(a) $x^3 + x^2 + x + 1, 2$
(b) $x^3 + x^2 - x + 1, 2$
(c) $x^3 + x^2 - x + 1, 3$
(d) None of these

Sol. (a) Using long division method,

$$
\begin{array}{l}
 x^3 + x^2 + x + 1 \\
x - 1 \,\overline{)\, x^4 + 1} \\
\quad - \; + \\
\hline
x^3 + 1 \\
x^3 - x^2 \\
\; - \; + \\
\hline
x^2 + 1 \\
x^2 - x \\
\quad x \\
\hline
x + 1 \\
x - 1 \\
- \; + \\
\hline
\quad 2
\end{array}
$$

Hence, quotient $= x^3 + x^2 + x + 1$ and remainder $= 2$

Linear Equations in One Variable

The expression of the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$, is a linear polynomial of one variable and equation involving only linear polynomial are called linear equations of one variable.

e.g. $5x + 8 = 9 - x$ is a linear equation in one variable.

- Graph of linear equation of one variable is a straight line, which is either parallel to the horizontal and vertical axis.

- Linear equation in one variable has unique solution.

In Two Variables

An equation of the form $ax + by + c = 0$, where $a, b, c, \in R, a \neq 0, b \neq 0$ and here x, y are variables is called a linear equation in two variables.

e. g. $2x + 3y = 5, \sqrt{2}x + \sqrt{3}y = 0$ are linear equations in two variables.
$2a + 3b = 0$

- The linear equation in two variables $ax + by + c = 0$ has an infinite number of solutions.
- The graph of equation $ax + by + c = 0$ is a straight line, so it is called as linear equation.
- Every point on graph of $ax + by + c = 0$ gives it solution.

Example 2: Solve $\frac{2}{x-3} + \frac{3}{x-4} = \frac{5}{x}$, where $x \neq 3, x \neq 4$ and $x \neq 0$

(a) $3\frac{1}{3}$
(b) $3\frac{1}{2}$
(c) $3\frac{1}{4}$
(d) None of these

Sol. (a) Given that, $\frac{2}{x-3} + \frac{3}{x-4} = \frac{5}{x}$

$$\Rightarrow \frac{2(x-4) + 3(x-3)}{(x-3)(x-4)} = \frac{5}{x}$$

$$\Rightarrow (5x - 17)x = 5(x^2 - 7x + 12)$$

$$\Rightarrow 5x^2 - 17x = 5x^2 - 35x + 60$$

$$\Rightarrow 18x = 60 \Rightarrow x = \frac{60}{18} = 3\frac{1}{3}$$

So, $x = 3\frac{1}{3}$ is a solution of the given equation.

Example 3: The length of a rectangle is 8 cm more than its breadth. If the perimeter of the rectangle is 68 cm, its length and breadth are respectively

(a) 21 cm, 13 cm

(b) 22 cm, 21 cm
(c) 23 cm, 13 cm
(d) 24 cm, 21 cm

Sol. (a) Let the breadth of the rectangle be x cm.

Then, its length $= (x + 8)$cm

$\therefore$ Perimeter of rectangle $= 2[x + (x + 8)]$

$$= 4x + 16$$

According to the given condition, $4x + 16 = 68$

$\Rightarrow\ 4x = 52 \Rightarrow x = 13$
$\therefore$ Breadth of rectangle $= 13$ cm
and length of rectangle $= 13 + 8 = 21$ cm.

Consistency of the System of Linear Equations

A set of linear equations is said to be consistent, if there exists atleast one solution for these equations.
A set of linear equations is said to be inconsistent, if there is no solution for these equations.

Let us consider a system of two linear equations as shown.

$\boldsymbol{a_1x + b_1y + c_1 = 0}$ **and** $\boldsymbol{a_2x + b_2y + c_2 = 0.}$

Consistent System The above system will be consistent,

if $\boldsymbol{\frac{a_1}{a_2} \neq \frac{b_1}{b_2}}$ **or** $\boldsymbol{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$

- If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then system has unique solution and represents a pair of intersecting lines.
- If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then system has infinite solutions and represents overlapping lines

Inconsistent System: The given system will be inconsistent, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ and do not have any solution.

Hence, it represents a pair of parallel lines.

Example 4: Which option is correct, for the following pair of equations?

$$x + 2y - 4 = 0 \text{ and } 3x + 6y - 12 = 0$$

(a) Consistent
(b) Consistent (dependent)
(c) Inconsistent
(d) None of these

Sol. (b) Given, pair of linear equations is

$$x + 2y - 4 = 0 \text{ and } 3x + 6y - 12 = 0$$

On comparing with standard form of pair of linear equations,

we get

$$a_1 = 1, b_1 = 2, c_1 = -4$$

and $a_2 = 3, b_2 = 6, c_2 =$

Now, $\frac{a_1}{a_2} = \frac{1}{3}, \frac{b_1}{b_2} = \frac{2}{6} = \frac{1}{3}$

and $\frac{c_1}{c_2} = \frac{-4}{-12} = \frac{1}{3}$

Clearly, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{3}$

Hence, the given pair of linear equations is consistent (dependent).

Factor and Factorization

- A polynomial $g(x)$ is called a factor of polynomial $p(x)$, if $g(x)$ divides $p(x)$ exactly.
- To express polynomial as the product of polynomials of degree less than that of the given polynomial is called as factorization.

Factorization by Common Factors

A factor which occurs in each term, is called the common factor.

e.g. Factorize $16x^2y + 4xy$

We have, $16x^2y = 2 \times 2 \times 2 \times 2 \times x \times x \times y$ and $4xy = 2 \times 2 \times x \times y$

Here, $2 \times 2 \times x \times y$ is common in these two terms.

Factorization by Splitting Middle Term

Let factors of the quadratic polynomial $ax^2 + bx + c$ be $(px + q)$ and $(rx + s)$.

Then, $ax^2 + bx + c = (px + q)(rx + s)$

$$= prx^2 + (ps + qr)x + qs$$

On comparing the coefficients of x^2, x and constant terms from both sides, we get $a = pr, b = ps + qr$ and $c = qs$.

Here, b is the sum of two numbers ps and qr, whose product is $(ps)(qr) = (pr)(qs) = ac$.

Thus, to factorize $ax^2 + bx + c$, write b as the sum of two numbers, whose product is ac.

Note To factorize $ax^2 + bx - c$ and $ax^2 - bx - c$, write b as the difference of two numbers whose product is $(-ac)$.

Example 5: Factors of $2x^2 + 7x + 3$ are

(a) $(x + 2)(x + 1)$
(b) $(2x + 1)(x + 3)$
(c) $(x + 3)(2x - 1)$
(d) $(2x - 2)(x - 3)$

Sol. (b) Given polynomial is $2x^2 + 7x + 3$

On comparing with $ax^2 + bx + c$, we get

$$\begin{array}{lll} & a & = 2, b = 7 \\ \text{and} & c & = 3 \\ \text{Now,} & ac & = 2 \times 3 = 6 \end{array}$$

So, all possible pairs of factors of 6 are 1 and 6, 2 and 3.

Clearly, pair 1 and 6 gives

$$1 + 6 = 7 = b$$

$$\therefore\ 2x^2 + 7x + 3 = 2x^2 + (1+6)x + 3$$

$$= 2x^2 + x + 6x + 3 = x(2x+1) + 3(2x+1)$$

$$= (2x+1)(x+3)$$

Factorization by Algebraic Identities

Sometimes, we do a factorization with the help of algebraic identities, which are given below.

1. $(a^2 - b^2) = (a+b)(a-b)$
2. $(a+b)^2 = a^2 + b^2 + 2ab$ and $(a-b)^2 = a^2 + b^2 - 2ab$
3. $(a+b)^2 - (a-b)^2 = 4ab$ and $(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$
4. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
5. $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
6. $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$
7. $(a^3 + b^3) = (a+b)(a^2 + b^2 - ab)$
8. $(a^3 - b^3) = (a-b)(a^2 + b^2 + ab)$
9. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac)$
10. If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

Example 6: Factorize $8a^3 - 343b^3$

(a) $(2a + 7b)(4a^2 + 14ab + 49b^2)$
(b) $(2a - 7b)(4a^2 + 14ab + 49b^2)$
(c) $(2a - 7b)(4a^2 - 14ab + 49b^2)$
(d) None of the above

Sol. (**b**)

$$8a^3 - 343b^3 = (2a)^3 - (7b)^3$$

$$= (2a - 7b)[(2a)^2 + (2a)(7b) + (7b)^2]$$

$$= (2a - 7b)(4a^2 + 14ab + 49b^2)$$

Factorization by Using Theorems

1. Remainder Theorem

- Let $p(x)$ be the polynomial in x of degree not less than one and α be a real number.
- If $p(x)$ is divided by $(x-\alpha)$, then remainder is $f(\alpha)$.

Note Remainder can be evaluated by substituting, $\mathrm{x} = \alpha$ in $\mathrm{p(x)}$.

2. Factor Theorem

- Let $p(x)$ be a polynomial in x of degree not less than one and α be a real number.
- If $p(\alpha) = 0$, then $(x-\alpha)$ is factor of $p(x)$.

Note If $(x-\alpha)$ is a factor of $p(x)$, then $p(\alpha) = 0$.

Example 7: The value of p, if $(2x-1)$ is a factor of $2x^3 + px^2 + 11x + p + 3$, is

(a) -7
(b) 7
(c) -6
(d) 5

Sol. (a) Let $q(x) = 2x^3 + px^2 + 11x + p + 3$

If $q(x)$ is divisible by $2x-1$, then $(2x-1)$ is a factor of $q(x)$.

Consider $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

On putting $x = \frac{1}{2}$ in $q(x)$, we have

$$q\left(\frac{1}{2}\right) = 2 \times \left(\frac{1}{2}\right)^3 + p\left(\frac{1}{2}\right)^2 + 11\left(\frac{1}{2}\right) + p + 3 = 0$$

$$\Rightarrow 2 \times \frac{1}{8} + p \times \frac{1}{4} + \frac{11}{2} + p + 3 = 0$$

$$\Rightarrow \frac{1}{4} + \frac{p}{4} + \frac{11}{2} + p + 3 = 0$$

$$\Rightarrow \frac{1 + p + 22 + 4p + 12}{4} = 0$$

$\Rightarrow 5p + 35 = 0$

$\Rightarrow 5p = -35$

$\therefore\ p = -7$

Quadratic Equation

- The Second Degree equation of polynomial is called quadratic equation.
- The general quadratic equation is given by $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

Roots of a Quadratic Equation

A value of a variable which satisfies the particular quadratic equation is called root of that equation or solution of the equation.

e.g. Let the equation is $x^2 - 6x + 8 = 0$.

Here, we take $x = 2$, then $2^2 - 6(2) + 8 = 0$

So, $x = 2$ is a root of the quadratic equation.

Solution of a Quadratic Equation

The solution of a quadratic equation can be Find by **two methods:**

1. By Factorization Method Let the quadratic equation be $ax^2 + bx + c = 0$. If the factors of $ax^2 + bx + c$ are $(x + \alpha)(x + \beta)$, then the solution is $x = -\alpha, -\beta$.

2. By Quadratic Formula If given equation is $ax^2 + bx + c = 0$, then roots of a quadratic equation can be determined by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $(b^2 - 4ac)$ is the discriminant (4) of the equation.

Nature of Roots of a Quadratic Equation

Let $D = b^2 - 4ac$ be the discriminant of the quadratic equation $ax^2 + bx + c = 0$.

1. If $D > 0$, then the two roots are real and unequal.

2. If $D = 0$, then the two roots are real and equal.

3. If $D < 0$, then there are no real roots.

4. If $D > 0$ and D is perfect square, then roots are rational.

5. If $D > 0$ and D is not a perfect square, then roots are irrational.

Note:
If one of the roots of the quadratic equation is $a + \sqrt{b}$, then its another root will be $a - \sqrt{b}$.

Sum and Products of the Roots

Let α, β be the roots of the equation

$\boldsymbol{ax^2 + bx + c = 0}$

1. The sum of the roots, $\alpha + \beta = -\frac{b}{a}$

2. The product of the roots, $\alpha \cdot \beta = \frac{c}{a}$

Formation of a Quadratic Equation

If the roots of equation are given to us say α and β, then $S =$ Sum of roots $= \alpha + \beta$

and $P =$ Product of roots $= \alpha\beta$

$\therefore$ The quadratic equation will be

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ or $x^2 - Sx + P = 0$

Example 8: Solve the equation $2x^2 + 14x + 9 = 0$.

(a) $\frac{-7+\sqrt{31}}{2}$
(b) $\frac{-7-\sqrt{31}}{2}$
(c) $\frac{-7+\sqrt{31}}{2}, \frac{-7-\sqrt{31}}{2}$
(d) None of these

Sol. (c) Given equation is $2x^2 + 14x + 9 = 0$.

Then,

$$x = \frac{-14 \pm \sqrt{(14)^2 - 4(2)(9)}}{2(2)}$$
$$= \frac{-14 \pm \sqrt{196 - 72}}{4} = \frac{-14 \pm \sqrt{124}}{4}$$
$$= \frac{-14 \pm 2\sqrt{31}}{4} = \frac{-7 \pm \sqrt{31}}{2}$$

$\therefore$ The roots are $\frac{-7+\sqrt{31}}{2}$ and $\frac{-7-\sqrt{31}}{2}$.

Example 9: If $\left(a - \frac{1}{a}\right) = 6$, then $(a^4 + 1/a^4) = ?$

(a) 1444
(b) 38
(c) 34
(d) 1442

Sol. (d) $a - \frac{1}{a} = 6$

On squaring both side $\left(a - \frac{1}{a}\right)^2 = (6)^2$

$$\Rightarrow \quad a^2 + \frac{1}{a^2} - 2 = 36$$
$$\Rightarrow \quad a^2 + \frac{1}{a^2} = 38$$

After squaring both side again

$$\Rightarrow \left(a^2 + \frac{1}{a^2}\right)^2 = 38^2$$
$$\Rightarrow a^4 + \frac{1}{a^4} + 2 = 1444$$
$$\Rightarrow a^4 + \frac{1}{a^4} = 1442$$

Example 10: Divide 36 into two parts, such that 5 times of the first part is more than 8 times of the second part by 24?

(a) 20,16
(b) 24,12
(c) 26,10
(d) 22,14

Sol. (b) Let two parts are ' x ' and ' y '

then $5x - 8y = 24$
and $x + y = 36$

On multiplying with 8 both sides, $8x + 8y = 288$

Adding in Eq. (i)

$13x = 288 + 24 \Rightarrow 13x = 312$
$x = 24$ and $y = 36 - 24 = 12$